

Then $\angle QSA = \angle Q'QR = \frac{\pi}{2}$. Since O is the circumcenter of isosceles triangle $Q'QS$,

$$\angle SQQ' = 2\angle OQQ' = 2\angle OBQ' = \angle QBQ'. \quad (2)$$

Let M be midpoint of QQ' . Points Q, M, S, A lie on the circle with diameter QA , because $\angle QMA = \frac{\pi}{2} = \angle QSA$. Thus

$$\angle SQQ' = \angle SQM = \angle SAM. \quad (3)$$

Observe that $\sin \angle SAQ = \frac{SQ}{AQ} = \frac{Q'Q}{AQ} = \frac{2MQ}{AQ} = 2 \sin \angle MAQ = 2 \sin \angle OAQ$. From that we get

$$\angle SAQ = \arcsin(2 \sin \angle OAQ). \quad (4)$$

From $\angle OAQ = \frac{\angle QAB - \angle QAC}{2}$ and equations (2) through (4),

$$\angle QBQ' = \angle SQQ' = \angle SAM = \angle SAQ - \angle OAQ,$$

which is equation (1), as claimed.

For the inequality on the right, simply note that we have proved that the middle difference is the maximum of $|\angle PCB - \angle PBC|$ over all points $P \in \ell$, while Problem 2255 established that this difference is at most $|\angle PAB - \angle PAC|$. This observation concludes the proof.

Also solved by the proposer; no solution was published before now.

For an alternative proof of the right inequality, let $x = |\angle PAB - \angle PAC|$, $0 \leq x < \frac{\pi}{3}$.

The inequality to prove reduces to $\arcsin\left(2 \sin \frac{x}{2}\right) \leq \frac{3x}{2}$, for $0 \leq x < \frac{\pi}{3}$, which is an elementary exercise. It is interesting to note that according to the solution of Problem 2255, the inequality there, namely $|\angle PAB - \angle PAC| \geq |\angle PCB - \angle PBC|$, holds for all isosceles triangles ABC for which $\angle A \geq \frac{\pi}{3}$ (and $\angle B = \angle C \leq \frac{\pi}{3}$), while the inequality fails for some positions of P in isosceles triangles with $\angle A < \frac{\pi}{3}$. Note that $\arcsin\left(2 \sin \frac{x}{2}\right)$ is no longer real for $x > \frac{\pi}{3}$, so that there are positions of P for which the right inequality of the present problem fails for isosceles triangles with $\angle A > \frac{\pi}{3}$.

3641. [2011 : 234, 237] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $0 \leq x_1, x_2, \dots, x_n < \pi/2$ be real numbers. Prove that

$$\left(\frac{1}{n} \sum_{k=1}^n \sec(x_k) \right) \left(1 - \left(\frac{1}{n} \sum_{k=1}^n \sin(x_k) \right)^2 \right)^{1/2} \geq 1.$$

I. Composite of similar solutions by Arkady Alt, San Jose, CA, USA; and Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

Let $f(x) = \sec x$, $g(x) = \sin x$ and set $\bar{x} = \frac{1}{n} \sum_{k=1}^n x_k$. Since $f''(x) = \frac{1 + \sin^2 x}{\cos^3 x} > 0$ and $g''(x) = -\sin x < 0$ for $0 < x < \frac{\pi}{2}$, f is convex and g is concave on the interval $(0, 1)$.

Hence Jensen's Inequality ensures that

$$\frac{1}{n} \sum_{k=1}^n \sec x_k \geq \sec(\bar{x}) \quad \text{and} \quad \frac{1}{n} \sum_{k=1}^n \sin x_k \leq \sin(\bar{x}).$$

Therefore we have

$$\begin{aligned} \left(\frac{1}{n} \sum_{k=1}^n \sec(x_k) \right) \left(1 - \left(\frac{1}{n} \sum_{k=1}^n \sin(x_k) \right)^2 \right)^{1/2} &\geq \sec(\bar{x})(1 - \sin^2(\bar{x}))^{1/2} \\ &= \sec(\bar{x}) \cos(\bar{x}) = 1. \end{aligned}$$

II. Composite of virtually identical solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and Salem Malikić, student, Simon Fraser University, Burnaby, BC.

By Cauchy-Schwarz Inequality we have

$$n \left(\sum_{k=1}^n \sin^2(x_k) \right) = \left(\sum_{k=1}^n 1^2 \right) \left(\sum_{k=1}^n \sin^2(x_k) \right) \geq \left(\sum_{k=1}^n \sin(x_k) \right)^2$$

so

$$\left(\frac{1}{n} \sum_{k=1}^n \sin(x_k) \right)^2 \leq \frac{1}{n} \sum_{k=1}^n \sin^2(x_k).$$

Hence,

$$\begin{aligned} \left(\frac{1}{n} \sum_{k=1}^n \sec(x_k) \right) \left(1 - \left(\frac{1}{n} \sum_{k=1}^n \sin(x_k) \right)^2 \right)^{1/2} &\geq \left(\frac{1}{n} \sum_{k=1}^n \sec(x_k) \right) \left(1 - \frac{1}{n} \sum_{k=1}^n \sin^2(x_k) \right)^{1/2} \\ &= \left(\frac{1}{n} \sum_{k=1}^n \sec(x_k) \right) \left(\frac{1}{n} \sum_{k=1}^n (1 - \sin^2(x_k)) \right)^{1/2} \\ &= \left(\frac{1}{n} \sum_{k=1}^n \frac{1}{\cos(x_k)} \right) \left(\frac{1}{n} \sum_{k=1}^n \cos^2(x_k) \right)^{1/2} \\ &\geq \left(\prod_{k=1}^n \frac{1}{\cos(x_k)} \right)^{1/n} \left(\left(\prod_{k=1}^n \cos^2(x_k) \right)^{1/n} \right)^{1/2} = 1 \end{aligned}$$

by the AM-GM Inequality.

Clearly, equality holds if and only if $x_1 = x_2 = \dots = x_n$.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; ALBERT STADLER, Herrlberg, Switzerland; and the proposer.